Partitioned Solution Procedure for Control-Structure Interaction Simulations

K. C. Park*
University of Colorado, Boulder, Colorado 80309
and
W. Keith Belvin†
NASA Langley Research Center, Hampton, Virginia 23665

A partitioned computational procedure is presented for the simulation of control-structure interaction systems by employing three modular software packages: a second-order structural dynamics analyzer, a second-order observer module, and a first-order stabilized active control force generator. This paper focuses on modular programming of the procedure, techniques for enhancing accuracy and for stabilizing the partitioned interaction equations, and time discretization of the stabilized systems of equations. A stability analysis has been performed using a set of model interaction equations, which indicates that the computational stability of the procedure is governed by the highest frequency of the controller and the strength of its position feedback parameter and not by that of the structural system. Comparison of the computational efficiency of the present procedure with a first-order conventional solution procedure indicates that the present procedure offers a substantial efficiency improvement. The effectiveness of the present procedure has been demonstrated by several example problems.

I. Introduction

REALISTIC analysis of active control-structure interaction (CSI) problems poses a formidable challenge both to the dynamist and the control scientist. The emerging need for real-time simulation and control of CSI systems will engender a new activity called "computational controls" to meet this challenge. The prevailing practice in CSI simulations is by and large limited to a reduced-order model of structures based on modal representations together with a limited number of sensors and actuators.[1-6] Specifically, the control laws are expressed in terms of the generalized coordinates and their time derivatives, which are computed using state estimators if full state feedback control is used. The resulting closed-loop system equations are usually cast in first-order form; this has been the prevailing practice since modern control theory has been almost exclusively developed for the first-order form.[7-10]

Typically, simulation tasks for CSI problems involve several computational elements and discipline-oriented models such as structural dynamics, control law synthesis, state estimation, actuator and sensor dynamics, thermal analysis, liquid sloshing and swirling, environmental disturbances, and maneuvering thrusts and torques. Because each of these computational elements can be large, it is usually not practical to assemble these computational elements into a single set of equations of motion and perform the analysis in its totality, which will be referred to as the "simultaneous solution approach." First, the equation size of the total system can be simply too large for many existing computers. Second, the solution of the coupled interaction equations may destroy the sparsity of the attendant matrices, thus requiring excessive computations and storage space. Most important of all, any changes in the model or in the computational procedures will affect many of the required analysis software modules and hence require a painstaking software verification effort.

The computational procedure that is described in the present paper has been motivated to alleviate the aforementioned difficulties that exist in the simultaneous solution approach. First, software development of any new capability is costly and time consuming; thus, if at all possible, it is preferable to use existing single-field analysis modules to conduct the coupled-field interaction analysis. Second, the tasks for model generation and methods development of each field are best accomplished by relying on the experts of each single-field discipline. To accommodate both the software considerations and the single-field expertise, a partitioned (or divide-and-conquer) analysis procedure is proposed for control-structure interaction analysis for direct output feedback systems.[11,12] The procedure abandons the conventional way of treating the CSI problems as one entity. Instead, it treats the structure (or plant), the observer, and the controller/observer interaction terms as separate entities. Thus, the CSI problem is recognized as a coupled-field problem, and a divide-and-conquer strategy is adopted for the development of a real-time computational procedure. A simular concept has been successfully applied to other interaction analyses such as fluid-structure interactions,[13] multistructural interaction systems,[14] Earth dam and pore-fluid interactions,[15] and multibody systems with constraints.[16]

The proposed partitioned analysis procedure hinges on three software and computational aspects. First, because large-scale simulation of interdisciplinary problems must rely on an efficient and versatile data management system, the data manager is used to handle the necessary interprocessor communications among the single-discipline analysis modules. Second, at each discrete time increment, the equations of motion for each discipline are solved separately by considering the interaction terms as external disturbances or applied forces. Third, when necessary, computational stabilization and accuracy improvements are introduced through augmentations and/or equation modifications. It is important to note that such partitioned solutions of each discipline equations can be carried out either on a sequential or parallel machine

Received Feb. 3, 1989; presented as Paper 89-1238 at the 30th Structures, Structural Dynamics, and Materials Conference, Mobile, AL, April 3–5, 1989; revision received Sept. 18, 1989.

*Professor of Aerospace Engineering, Center for Space Structures and Control. Member AIAA.

†Aerospace Engineer, Spacecraft Dynamics Branch. Member AIAA.

if certain message passing and memory-conflict issues are handled appropriately.

Recently, the computational need and the physical insight that can be accrued from the second-order dynamic equations have motivated several investigators in the control community to address the second-order state estimation and control law issues. They include the analysis of the second-order system equations[17,18] and CSI design,[19–26] among others.

The objective of the paper is thus to describe the algorithmic nature of the partitioned CSI solution procedure, its implementation aspects and computational stability and accuracy characteristics. To this end, the paper is organized as follows. The discrete equations of motion for CSI problems are presented in Sec. II, which includes the discrete equations of motion for structures subjected to control forces, an observer model, and a general control law.

A review of a conventional first-order-based solution procedure is presented in Sec. III, which is termed herein simultaneous solution procedure to contrast with the proposed partitioned procedure. Section IV introduces a parabolic stabilization in order to derive a differential equation for the interaction terms (i.e., the control force and the state estimation error term) so that the interaction terms are obtained by solving differential equations rather than by back substitutions. It turns out that such a stabilization (or an equation augmentation) improves not only computational stability but numerical accuracy as well.

Time discretization of the coupled CSI equations is carried out in Sec. V by employing the implicit midpoint formula (or the trapezoidal rule). It is shown that a general canonical form of the equations of motion for structures and the second-order form of state estimators leads to a computationally attractive discretization. A computational stability analysis of the present partitioned CSI solution procedure is elaborated in Sec. VI, which indicates that the stability of the present partitioned procedure is governed by the highest frequency of the control force and its position feedback strength parameter irrespective of the frequency magnitude of the structural system.

The computational efficiency of the present procedure is compared with the conventional procedure in Sec. VII. It is shown that the present procedure offers a substantial efficiency improvement over the conventional procedure for most practical CSI problems. Applications of the present procedure are given by way of several example analyses; results are offered in Sec. VIII. Finally, concluding remarks and further remaining challenges toward making the real-time CSI analysis and simulations a routine practice are discussed in Sec. IX.

II. Equations of Motion for Control-Structure Interaction Systems

A typical control-structure interaction system can be represented as shown in Fig. 1 (see, e.g., Kwakernaak and Sivan[9]). The discrete equations of motion for control-structure interaction systems may be described by

Structure:

$$M\ddot{q} + D\dot{q} + Kq = f + Bu + Gw$$

$$q(0) = q_0, \qquad \dot{q}(0) = \dot{q}_0 \tag{1a}$$

Sensor output:

$$z = Hx + v \tag{1b}$$

Estimator:

$$\dot{\tilde{x}} = A\tilde{x} + Ef + \bar{B}u + L\gamma, \qquad \tilde{x}(0) = 0 \tag{1c}$$

Control force:

$$u = -F\tilde{x} \tag{1d}$$

Estimation error:

$$\gamma = z - (H_d\tilde{q} + H_v\dot{\tilde{q}}) \tag{1e}$$

where

$$x = \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix}, \qquad \tilde{x} = \begin{Bmatrix} \tilde{q} \\ \dot{\tilde{q}} \end{Bmatrix}$$

and

$$H = [H_d \quad H_v], \qquad L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \qquad E = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ M^{-1}B \end{bmatrix}, \quad F = [F_1 \quad F_2]$$

In the preceding equations, M is the mass matrix, D is the damping matrix, K is the stiffness matrix, $f(t)$ is the applied force, B is the actuator location matrix, G is the disturbance location matrix, q is the generalized displacement vector, w is a disturbance vector, and the superscript dot denotes time differentiation. In Eq. (1b), z is the measured sensor output. The matrix H_d is the matrix of displacement sensor locations and H_v is the matrix of velocity sensor locations. The vector v is measurement noise. The state estimator in Eq. (1c) is assumed either to be based on the Kalman filter[7,8] or based on a Luenberger observer[27] if the system is deterministic. The superscipt ~ denotes the estimated states. The actuator output u is a function of the state estimator variables $\tilde{q}$ and $\dot{\tilde{q}}$, and F_1 and F_2 are control gains determined for example by pole-zero placement or from the solution of an optimal control problem. The observer is governed by L, the filter gain matrix. For the special case where L_1 is the null matrix (i.e., $\dot{\tilde{q}} = \tilde{\dot{q}}$), a second-order state estimator can be expressed as

$$M\ddot{\tilde{q}} + D\dot{\tilde{q}} + K\tilde{q} = f + Bu + ML_2\gamma \tag{2}$$

The effect of the preceding simplification on the observer stability and convergence is discussed in detail in Refs. 26 and 28.

III. Simultaneous Solution Approach

It should be emphasized that the objective of this paper is to directly solve the coupled equations given in Eqs. (1) in their natural forms. That is, to exploit the second-order form of the structure equation (and also the observer equation if $L_1 = 0$). To justify this objective, the simultaneous solution approach for numerical simulation of Eq. (1) is examined first.

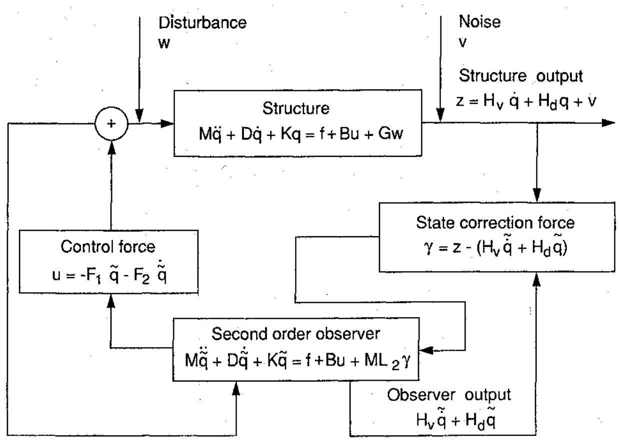

Fig. 1 Typical control/structure interaction system.

The numerical solution of Eqs. (1) by the simultaneous solution approach begins with appropriate initial conditions: the feedback gain F and the filter gain L. The structure equation is written in first-order form

$$\dot{x} = Ax + Ef + \bar{B}u + \bar{G}w \tag{3}$$

where

$$\bar{G} = \begin{bmatrix} 0 \\ M^{-1}G \end{bmatrix}$$

The control gains and observer gains can be synthesized independently by noting that the stability of the structural system and the observer error stability are uncoupled. Introducing the error equation by the deterministic form of Eqs. (1) as

$$\bar{e} = x - \tilde{x} = \begin{Bmatrix} q - \tilde{q} \\ \dot{q} - \dot{\tilde{q}} \end{Bmatrix} \tag{4}$$

and eliminating u yields

$$\begin{Bmatrix} \dot{x} \\ \dot{\bar{e}} \end{Bmatrix} = \begin{bmatrix} A - \bar{B}F & \bar{B}F \\ 0 & A - LH \end{bmatrix} \begin{Bmatrix} x \\ \bar{e} \end{Bmatrix} + \begin{bmatrix} E \\ 0 \end{bmatrix} f + \begin{bmatrix} \bar{G} \\ 0 \end{bmatrix} w \tag{5}$$

The stability of Eq. (5) is governed by the stability of $[A - \bar{B}F]$ and $[A - LH]$. Thus, the control gain F is suitably chosen from the matrix $[A - \bar{B}F]$ and the observer gain L from the matrix $[A - LH]$.

Subsequently, the simultaneous solution approach eliminates u and z from Eqs. (1a) and (1c) and then solves the observer-based closed-loop equations:

$$\begin{Bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{Bmatrix} = \begin{bmatrix} A & -\bar{B}F \\ LH & A - \bar{B}F - LH \end{bmatrix} \begin{Bmatrix} x \\ \tilde{x} \end{Bmatrix} + \begin{Bmatrix} E \\ E \end{Bmatrix} f + \begin{Bmatrix} \bar{G} \\ 0 \end{Bmatrix} w \tag{6}$$

The embedding effects of both the controller and the state observer result in an unsymmetric and nonsparse system matrix of dimension $(4N \times 4N)$, where N is the number of structural degrees of freedom. Solution of Eq. (6) would require considerable software modification of existing structural dynamics analysis programs for large-scale CSI simulation purposes. In addition to losing the computational advantages associated with the finite element based CSI equation, the simultaneous solution approach requires the control law to be embedded into the observer model. If the control law includes actuator, sensor, and/or controller dynamics, additional states must be added to the observer. This greatly complicates the observer model and requires significant software development for each class of control law dynamics. The difficulties associated with the simultaneous solution approach have prompted development of a partitioned solution approach for the CSI equations as described in the next section.

IV. Partitioned Solution Procedure

To mitigate the software and algorithmic difficulties associated with the asymmetric embedding of the controller and the state observer into the closed-loop equations as discussed in the previous section, a preliminary partitioned solution procedure[12] was proposed for the solution of direct output feedback systems, but no stability analysis was offered therein. The present exposition extends the basic concepts offered in Ref. 12 to include the case of dynamic compensators in a form of computationally advantageous second-order observers. However, for completeness, the first-order observer solution procedure is included in the Appendix.

In essence, the proposed partitioned solution procedure numerically integrates the structural equations of motion (1a) and the observer equation (1c) by treating the control force u and the estimation error γ as if they are applied terms in the right-hand sides. In this way, simulation of control-structure interaction systems using the proposed partitioned solution procedure to be described can be carried out by a judicious employment of three software modules: the structural analyzer to obtain q, the state estimator to obtain $\tilde{q}$, and the solver for the control force u and the state estimation error γ as indicated in Fig. 2. Thus, the partitioned procedure becomes computationally efficient and can preserve software modularity by exploiting the symmetric matrix form in the left-hand sides of Eqs. (1a) and (1c).

However, computations of the control force u and the state estimation error γ by Eqs. (1d) and (1e), respectively, can lead not only to an accumulation of errors but often can give rise to numerical instability. Hence, to make the proposed partitioned solution procedure robust, it is imperative to stabilize the partitioned solution process and/or numerically to filter the solution errors in computing u and γ. This is addressed in the next section.

Stabilization for Computations of Control Force and Estimation Error

To appreciate the nature of error accumulations in computing u and γ by employing Eqs. (1d) and (1e), respectively, let us consider the following discrete predictors:

$$u^{n+1/2} \approx u^n = -(F_1\tilde{q}^n + F_2\dot{\tilde{q}}^n) \tag{7a}$$

$$\gamma^{n+1/2} \approx \gamma^n = z^n - (H_d\tilde{q}^n + H_v\dot{\tilde{q}}^n) \tag{7b}$$

where the superscripts designate the discrete time intervals, $t^n = nh$ and $t^{n+1/2} = n + 1/2h$ with a constant stepsize of h.

It is observed that the predicted control force $u^{n+1/2}$ will possess errors in $(\tilde{q}^{n+1/2} - \tilde{q}^n)$ and $(\dot{\tilde{q}}^{n+1/2} - \dot{\tilde{q}}^n)$ magnified by F_1 and F_2, unless the control gains have a built-in stabilizing filter against these errors. The computation of the estimation error $\gamma^{n+1/2}$ will have a similar error accumulation unless the filter gain L is designed to cope with the prediction errors. Among several possible stabilization strategies, we will employ a parabolic stabilization as follows.

First, we time differentiate Eq. (1c) to obtain

$$\dot{u} = -F_1\dot{\tilde{q}} - F_2\ddot{\tilde{q}} \tag{8}$$

Substituting $\ddot{\tilde{q}}$ from Eq. (2) into the preceding equation, one obtains

$$\dot{u} + F_2M^{-1}Bu = -F_2(M^{-1}\dot{\tilde{p}} + L_2\gamma) - F_1\dot{\tilde{q}} \tag{9}$$

where the generalized rate of momentum $\dot{\tilde{p}}$ is given by

$$\dot{\tilde{p}} = (f - D\dot{\tilde{q}} - K\tilde{q}) \tag{10}$$

The parabolic stabilization that led to Eq. (9) for computing the control law is sometimes called an equation augmentation procedure as it has not altered any of the basic governing of Eqs. (1) except one time differentiation of u assuming $\dot{u}$ exists. However, the assumption is later removed through time discretization as will be shown later in the paper.

It is noted that the homogeneous part of Eq. (9) has the filtering effect of the form $(sI + F_2M^{-1}B)^{-1}$ in parlance of classical control theory where s is the Laplace transform operator, thus achieving the required stabilization. From the computational viewpoint, although $F_2M^{-1}B$ is in general a full matrix, its size is relatively small as the size of u is proportional to the number of actuators placed on the structure.

Similarly, for the observer estimation error γ, one can stabilize its computation by adopting the following augmented equation:

$$\dot{\gamma} + H_vL_2\gamma = \dot{z} - H_vM^{-1}(\dot{\tilde{p}} + Bu) - H_d\dot{\tilde{q}} \tag{11}$$

Again the matrix H_vL_2 is in general not sparse; however its size is only proportional to the number of measurements output from the structure.

Stabilized Partitioned Equations and Solution Process

The adoption of the second-order observer and the preceding stabilization thus replace Eqs. (1c), (1d), and (1e) by Eqs. (2), (9), and (11), respectively, as summarized here.

Structure:

$$M\ddot{q} + D\dot{q} + Kq = f + Bu + Gw$$
$$q(0) = q_0, \qquad \dot{q}(0) = \dot{q}_0 \tag{12a}$$

Sensor output:

$$z = Hx + v \tag{12b}$$

Estimator:

$$M\ddot{\tilde{q}} + D\dot{\tilde{q}} + K\tilde{q} = f + Bu + ML_2\gamma$$
$$\tilde{q}(0) = 0, \qquad \dot{\tilde{q}}(0) = 0 \tag{12c}$$

Control force:

$$\dot{u} + F_2M^{-1}Bu = -F_2(M^{-1}\dot{\tilde{p}} + L_2\gamma) - F_1\dot{\tilde{q}} \tag{12d}$$

Estimation error:

$$\dot{\gamma} + H_vL_2\gamma = \dot{z} - H_vM^{-1}(\dot{\tilde{p}} + Bu) - H_d\dot{\tilde{q}} \tag{12e}$$

The partitioned solution process that implements the preceding equation can be summarized by the following steps:

1) Obtain $\tilde{q}_p^{n+1/2} = \tilde{q}^n + \delta\dot{\tilde{q}}^n$ and $\dot{\tilde{q}}_p^{n+1/2} = \dot{\tilde{q}}^n$.
2) Solve for $u^{n+1/2}$ and $\gamma^{n+1/2}$ from Eqs. (12d) and (12e).
3) Solve for $\tilde{q}^{n+1/2}$, $\tilde{q}^{n+1}$, and $\dot{\tilde{q}}^{n+1}$ from Eq. (12c).
4) Solve for $q^{n+1/2}$, q^{n+1}, and $\dot{q}^{n+1}$ from Eq. (12a).
5) Increment $(n = n + 1)$ and $(\text{time} = \text{time} + h)$.

A flowchart describing the preceding computational steps is depicted by Fig. 2. These computational steps are implemented into three modules as shown in Fig. 2. Note that steps 1–5 would be repeated until the transient response simulation is complete. (Note, if no observer is used, step 2 would only require calculation of $u^{n+1/2}$ and step 3 would be omitted.) The methods for calculating the control and observer interaction forces differ depending on the type of simulation required. This aspect together with the time discretization, prediction of the state estimation vectors, and computational details for implementing the present partitioned solution procedure will be dealt with in the next section.

V. Time Discretization of Control-Structure Interaction Equations

Direct time integration of the structural equations (12a) and the observer equations (12c) can be performed by a variety of techniques. A theoretically exact solution can be obtained using a matrix exponential approach. However, such an approach engenders numerical roundoff when applied to large-dimensioned problems and involves fully populated matrices that can significantly limit the number of simultaneous equations which can be solved. A number of approximate techniques for numerical integration of coupled ordinary differential equations also exist. The approximate methods generally interpolate and/or extrapolate the dynamic states with implicit and/or explicit time-stepping formulas. By preserving the sparsity and symmetry of the coefficient matrices, the approximate methods can be applied to very large systems of simultaneous equations. Thus approximate numerical integration formulas are used in this paper.

Approximate methods must be examined from both stability and accuracy considerations. Numerical stability of explicit integration formulas require a step size inversely proportional to the highest frequency of the system. The small step size necessary for explicit integration stability generally satisfies the accuracy requirements. Conversely, some fully implicit formulas are unconditionally stable; thus the step size must be chosen based on accuracy considerations. Because larger step sizes permit more efficient solution of the control-structure interaction equations, implicit integration formulas are employed in this study.

It should be mentioned that, when massively parallel computations become widely available, the programming simplicity of explicit formulas become so attractive that one may prefer explicit to implicit formulas for control-structure interaction simulations. This is particularly true when reduced-order models are used that filter high-frequency responses.

Among a plethora of implicit numerical integration formulas, we employ the following unconditionally stable midpoint implicit formulas:

$$q^{n+1/2} = q^n + \delta\dot{q}^{n+1/2}, \qquad \delta = h/2 \tag{13a}$$
$$\dot{q}^{n+1/2} = \dot{q}^n + \delta\ddot{q}^{n+1/2} \tag{13b}$$
$$q^{n+1} = 2q^{n+1/2} - q^n \tag{13c}$$

where h is the time-step size. The selection of the midpoint implicit formula (or the trapezoidal rule) for the present application is due to its minimal frequency distortion and no numerical damping characteristics.[29–31] Experience has shown that it is very important to minimize numerical damping when studying the control-structure interactions.

Integration of Structural Equations

Implicit time discretization of Eq. (12a) by Eq. (13) yields the following difference equation for the structure:

$$Sq^{n+1/2} = g^{n+1/2} \tag{14a}$$
$$S = M + \delta D + \delta^2 K \tag{14b}$$
$$g^{n+1/2} = \delta^2(f^{n+1/2} + Bu^{n+1/2}) + M(q^n + \delta\dot{q}^n) + \delta Dq^n + \delta^2 Gw^{n+1/2} \tag{14c}$$
$$q^{n+1} = 2q^{n+1/2} - q^n \tag{14d}$$
$$\dot{q}^{n+1/2} = (q^{n+1/2} - q^n)/\delta, \qquad \dot{q}^{n+1} = 2\dot{q}^{n+1/2} - \dot{q}^n \tag{14e}$$

It is observed that the only unknown in the right-hand side of the preceding difference equation is $u^{n+1/2}$, which is treated as an applied forcing term in solving for the states q^{n+1} and $\dot{q}^{n+1}$, thus, leading to a modular implementation as a unique feature in the present partitioned solution procedure.

Integration of Observer Equations

Time discretization of Eq. (12c) yields the following equations for the observer:

$$S\tilde{q}^{n+1/2} = \tilde{g}^{n+1/2} \tag{15a}$$

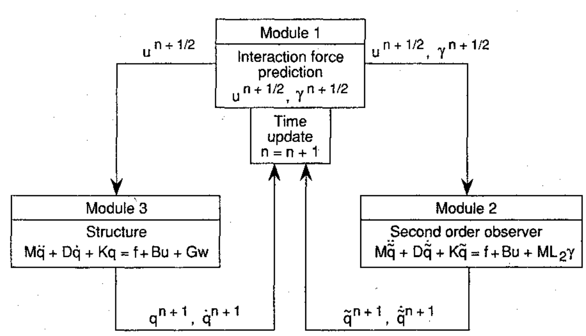

Fig. 2 Flow chart of partitioned solution procedure.

$$S = M + \delta D + \delta^2 K \tag{15b}$$

$$\tilde{g}^{n+1/2} = \delta^2(f^{n+1/2} + Bu^{n+1/2} + ML_2\gamma^{n+1/2}) + M(\tilde{q}^n + \delta\dot{\tilde{q}}^n) + \delta D\tilde{q}^n \tag{15c}$$

$$\tilde{q}^{n+1} = 2\tilde{q}^{n+1/2} - \tilde{q}^n \tag{15d}$$

$$\dot{\tilde{q}}^{n+1/2} = (\tilde{q}^{n+1/2} - \tilde{q}^n)/\delta, \qquad \dot{\tilde{q}}^{n+1} = 2\dot{\tilde{q}}^{n+1/2} - \dot{\tilde{q}}^n \tag{15e}$$

From Eqs. (15) it is seen that $u^{n+1/2}$ and $\gamma^{n+1/2}$ are required to numerically solve for the states $\tilde{q}^{n+1}$ and $\dot{\tilde{q}}^{n+1}$. As in the case of the structural analyzer of Eqs. (14), these two unknown vectors are treated as applied disturbance terms so that the observer states can be implemented into a modular package. Note that both in Eqs. (14) and (15) the attendant matrix S possesses the same sparsity and symmetry that exists in the mass, damping, and stiffness matrices. Hence, linear equation solution techniques that exploit symmetry and sparsity may be used advantageously. Such a solution capability is widely available in existing structural analysis software systems.

Integration of Control Force and State Estimation Error Equations

In advancing the time marching for the structural and the state estimation equations, it is recalled that we have assumed that both $u^{n+1/2}$ and $\gamma^{n+1/2}$ are already available from the solution module of the control force and state estimation error equations. We will show in this subsection that $u^{n+1/2}$ and $\gamma^{n+1/2}$ can be obtained by predicting only the state estimation vector $\tilde{q}^{n+1/2}$.

In doing so, unlike the structural and observer equations, we have decided to time discretize both the control force and state estimation error equations as a coupled set of equations. Carrying out the necessary discretization by the midpoint implicit formula [Eqs. (13)] yields

$$\bar{S}r^{n+1/2} = \bar{g}^{n+1/2} \tag{16a}$$

$$\bar{S} = \begin{bmatrix} I + \delta F_2 M^{-1}B & \delta F_2 L_2 \\ \delta H_v M^{-1}B & I + \delta H_v L_2 \end{bmatrix} \tag{16b}$$

$$\bar{g}^{n+1/2} = \begin{Bmatrix} 0 \\ z_p^{n+1/2} \end{Bmatrix} - \begin{Bmatrix} F_1 \\ H_d \end{Bmatrix}\tilde{q}_p^{n+1/2} - \begin{Bmatrix} F_2 \\ H_v \end{Bmatrix}(\dot{\tilde{q}}^n + M^{-1}\tilde{p}_p^{n+1/2}) \tag{16c}$$

$$r^{n+1/2} = \begin{Bmatrix} u^{n+1/2} \\ \gamma^{n+1/2} \end{Bmatrix} \tag{16d}$$

It is noted that $\bar{S}$ is not sparse; thus the partitioned solution approach is mainly oriented toward systems where the number of states ($2N$) is more than the number of actuators m and measurements r (i.e., $r + m < 2N$). The solution of Eqs. (16) at the half time step makes both $u^{n+1/2}$ and $\gamma^{n+1/2}$, thus permitting the solution of both the structural displacement and the state estimation vectors with the control and state estimation error terms on the right-hand sides of Eqs. (14) and (15). Since a prediction step is involved, one typically iterates to achieve a converged solution for $\tilde{q}^{n+1/2}$. However, as will be shown in Sec. VIII, the stabilization procedure used to obtain Eqs. (9) and (11) [or Eqs. (12d) and (12e)] produces very accurate results without iterating to obtain $\tilde{q}^{n+1/2} = \tilde{q}_p^{n+1/2}$ in most instances. To solve for $u^{n+1/2}$ when an observer is used, it is necessary to predict $\tilde{q}^{n+1/2}$

$$\tilde{q}_p^{n+1/2} = q^n + \delta\dot{\tilde{q}}^n \tag{17}$$

and use $z^{n+1/2}$ to solve Eq. (16) for $u^{n+1/2}$ and $\gamma^{n+1/2}$. (Note that $z^{n+1/2}$ implies measuring the system output at the half time step. If this is not possible, using $z_p^{n+1/2} = z^n$ will produce a phase shift in the observer state of δ in time.)

VI. Stability and Accuracy of Partitioned Solution Procedure

Computational stability analysis of partitioned procedures for a general coupled system is still in an evolving stage as discussed in Refs. 14 and 32. Hence, the analysis herein applies the relevant results from Refs. 14 and 32 in the present stability analysis of the partitioned CSI solution procedure. The partitioned CSI solution procedure presented in Eqs. (14–16), although discretized by implicit time integration formulas, may suffer from computational instability as it involves extrapolations to obtain $u^{n+1/2}$ and $\gamma^{n+1/2}$. A complete stability analysis of the partitioned solution procedure for the coupled structural dynamics, observer, and controller equations is difficult to perform unless the observer characteristics H,L and the controller characteristics B,F are specified. Hence, the analysis that follows is restricted to an ideal observer, i.e., $\gamma = 0$ so that only Eq. (14) and Eq. (16a) are considered.

Stability Analysis of Model Control-Structure Interaction

To assess the computational stability of the present partitioned solution procedure, we construct a model single-degree-of-freedom interaction equation as follows. First, neglecting structural damping, a modal structural equation of motion can be expressed as

$$\ddot{y} + \omega^2 y = -u \tag{18}$$

where y is a generalized coordinate and ω is its associated frequency.

Second, the model controller is assumed to consist of both the position and velocity feedback with appropriate weights given by

$$u = \eta\omega_c^2 y + \zeta\omega_c\dot{y}, \qquad \omega_{\min} \le \omega_c \le \omega_{\max} \tag{19}$$

where ω_c is the feedback frequency that ranges from the minimum to the maximum of the structural frequency contents, and η and ζ are positive scalar coefficients that signify the strength of the position and the velocity feedback, respectively.

Combining Eq. (18) with the stabilized form of Eq. (19), we have the model interaction equation

$$\ddot{y} + \omega^2 y = -u \tag{20a}$$

$$\dot{u} + \zeta\omega_c u = \eta\omega_c^2\dot{y} - \zeta\omega_c\omega^2 y \tag{20b}$$

Thus, the model interaction equations given by Eqs. (20) represent the case of full-state feedback. They do not, however, reflect the mode-to-mode coupling that can occur in reduced-order feedback controller. Nevertheless, an analysis of the computational stability using the preceding model interaction equations should shed insight on the overall stability of the present partitioned solution procedure.

Application of the partitioned CSI solution procedure [Eqs. (14–16)] with $\gamma = 0$ to solve the preceding model equations yields

$$y_p^{n+1/2} = y^n + \delta\dot{y}^n \tag{21a}$$

$$(1 + \delta\zeta\omega_c)u_p^{n+1/2} = (\eta\omega_c^2 - \delta\zeta\omega_c\omega^2)y_p^{n+1/2} + \zeta\omega_c\dot{y}^n \tag{21b}$$

$$(1 + \delta^2\omega^2)y^{n+1/2} = -\delta^2 u_p^{n+1/2} + y^n + \delta\dot{y}^n \tag{21c}$$

$$y^{n+1} = 2y^{n+1/2} - y^n, \qquad \dot{y}^{n+1/2} = (y^{n+1/2} - y^n)/\delta$$

$$\dot{y}^{n+1} = 2\dot{y}^{n+1/2} - \dot{y}^n \tag{21d}$$

$$u^{n+1} = \eta\omega_c^2 y^{n+1} + \zeta\omega_c\dot{y}^{n+1} \tag{21e}$$

Computational stability of the model form of CSI partitioned equations (20) can be assessed by seeking a nontrivial solution of the preceding difference equation (21) by

$$\begin{Bmatrix} u^{n+1} \\ y^{n+1} \end{Bmatrix} = \lambda \begin{Bmatrix} u^{n} \\ y^{n} \end{Bmatrix} \tag{22}$$

such that

$$|\lambda| \le 1 \tag{23}$$

for stability.

Substituting Eq. (22) into Eqs. (21) and eliminating $\dot{y}$, one obtains

$$J \begin{Bmatrix} u \\ y \end{Bmatrix}^n = 0 \tag{24}$$

where

$$J = \begin{bmatrix} \delta(1+\delta\zeta\omega_c)\cdot(\lambda+1)^2 & 4\lambda(\delta^2\zeta\omega_c\omega^2 - \delta\eta\omega_c^2) + 2\zeta\omega_c(1+\lambda) \\ \delta^2(\lambda+1)^2 & (1+\delta^2\omega^2)\cdot(\lambda+1)^2 - 4\lambda \end{bmatrix}$$

To test the stability requirement of Eq. (23) on the characteristic equation, i.e., $\det|J| = 0$, one transforms $|\lambda| \le 1$ into the entire left-hand plane of the z plane by

$$\lambda = \frac{1+z}{1-z}, \qquad |\lambda| \leqslant 1 \Leftrightarrow Re(z) \le 0 \tag{25}$$

Carrying out the necessary algebra we have from $\det|J(z)| = 0$ the following z-polynomial equation:

$$(\delta^3\zeta\omega_c\omega^2 - \delta^2\eta\omega_c^2 + 1)z^2 + (\delta\zeta\omega_c)z + \delta^2(\eta\omega_c^2 + \omega^2) = 0 \tag{26}$$

A test of the polynomial equation (26) for possible positive real roots by the Routh-Hurwitz criterion (see, e.g., Gantmacher[33]) indicates that the partitioned procedure as applied to the model coupled equations (18) and (19) gives a stable solution provided

$$(\delta^3\zeta\omega_c\omega^2 - \delta^2\eta\omega_c^2 + 1) \ge 0 \tag{27}$$

Note that if there is no position feedback (i.e., $\eta = 0$), the model interaction equations solved by the present partitioned solution procedure [Eqs. (14–16)] yields unconditionally stable solutions as Eq. (27) is automatically satisfied. Hence, a more critical stability assessment can be made by assuming no velocity feedback (i.e., $\zeta = 0$) for which we have stability from Eq. (27)

$$h \le 2/\sqrt{\eta}\omega_c \tag{28}$$

The preceding stability analysis on the model interaction equations permits us to make the following observations. First, Eq. (28) indicates that feedback frequency ω_c and the strength of the position feedback η dictate the computational stability and not the structural frequency ω. In other words, the position feedback dictates the allowable step size for stability. Thus the highest frequency of the controller governs stability, not the highest frequency of the structure. Since most controllers are designed with reduced-order structure models that ignore high-frequency dynamics, the present solution procedure is not unduly restricted by stability.

Second, if velocity feedback is present, the allowable step size for stability increases until $\zeta \ge \sqrt{4\eta^3/27}$ at which point the solution becomes unconditionally stable.

It should be noted that, instead of the stabilized form of control force [Eq. (12d) or (20b)], if the scalar form of Eq. (7a) is used in the preceding stability analysis, the resulting stability limit is given by

$$h \le \min(2/\zeta\omega_c, 2\zeta/\eta\omega_c) \tag{29}$$

Assuming $\zeta \ll 1$, the first term in the preceding condition allows a sufficiently large step size. However, since $\zeta/\eta \approx 1$ for a balanced control law, it imposes a step size restriction $h \approx 2/\omega_c$, which approaches the limit imposed by a typical explicit integration formula. This proves the advantage of the present stabilized partitioned solution equation (12) solely from the computational stability viewpoint.

Although not elaborated herein, a stability analysis that includes an observer model and the state estimation error equation has been conducted with the following parameter choices:

$$L_2 = \lfloor l_{21} \quad l_{22} \rfloor, \qquad H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{30}$$

in conjunction with the structural model already used in Eq. (20). The analysis result yields the following step size restriction

$$h \le \min(2/\zeta\omega_c, 2/\sqrt{\eta}\omega_c, 2/\sqrt{l_{21}}) \tag{31}$$

It should be noted that l_{21} corresponds to the Kalman filter gain magnitude, which can be adjusted to be sufficiently small compared with ω^2 as can be assessed from Eq. (5b). Hence, provided $l_{21} < \omega_c$, the condition given by Eq. (28) is seen to govern the maximum stable step size by the present partitioned solution procedure.

For the general multidimensional case governed by Eqs. (14–16), one observes that the stiffness proportional control force in practice reaches only a fraction of the total internal force ($\boldsymbol{u} = \eta K \boldsymbol{q}$, $\eta \ll 1$). Hence, even for a distributed stiffness proportional control configuration where $\omega_c \to \omega_{\max}$, the stable step size given by Eq. (28) should be much larger than the maximum stable step size of a typical explicit integration algorithm (say, $h_{\max} \le 2/\omega_{\max}$). Therefore, the computational efficiency of the present partitioned solution procedure is established.

Accuracy Assessment

In addition to the stability consideration, the step size must also be chosen based on accuracy considerations. Although the midpoint implicit integration formulas produce no artificial damping, there does exist a frequency distortion. The apparent frequency to true frequency ratio F_a is given by

$$F_a = \frac{N_s}{2\pi}\tan^{-1}\left(\frac{2\pi}{N_s[1 - (\pi^2/N_s^2)]}\right) \tag{32}$$

where N_s is the number of steps per cycle, $N_s = (2\pi)/(h\omega)$. Figure 3 shows the frequency distortion as a function of the number of steps per cycle. There must be 18 steps per cycle to produce less than 1% frequency error. Thus, the step size based on 1% frequency error is

$$h \le 2\pi/N_s\omega = 0.349/\omega \tag{33}$$

Comparing Eq. (28) to Eq. (33) one finds $\eta \ge 32.84$ before stability governs the step size and not the 1% allowable frequency distortion. Thus, for most practical considerations, accuracy of the present integration formulas rather than stability govern the selection of the integration step size h.

VII. Computational Efficiency

Simulation of the CSI equations in second-order form is prompted in part by potential increases in computational

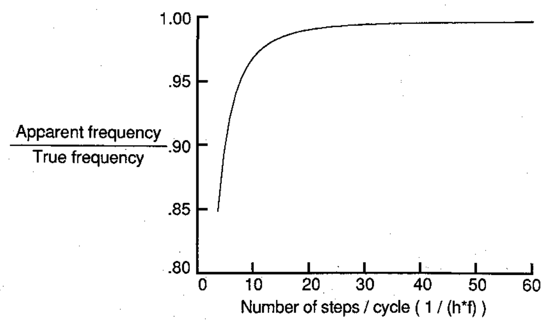

Fig. 3 Frequency distortion using midpoint implicit integration.

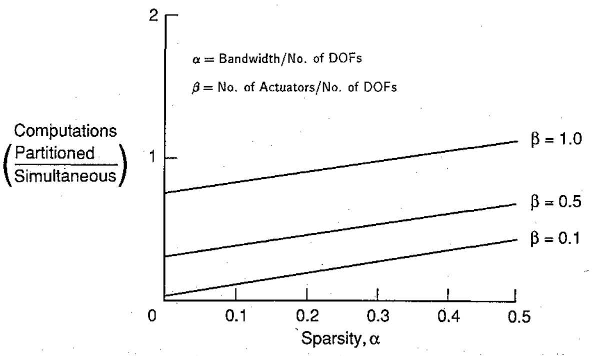

Fig. 4 Computational efficiency of partitioned solution procedure.

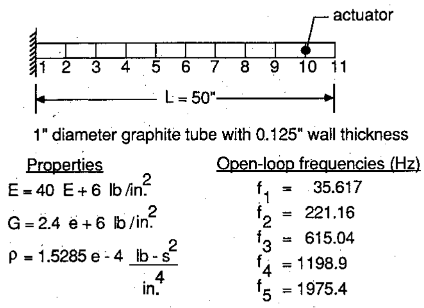

Fig. 5 Cantilever beam example.

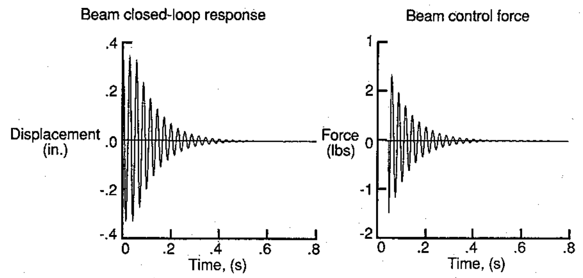

Fig. 6 Closed-loop response of example beam.

efficiency. The computational benefits of matrix sparsity and symmetry on the left-hand side (LHS) of the second-order finite element based equations are reduced by the additional computations needed to maintain the control and state estimation error terms on the right-hand side (RHS) of the equations. Therefore, we will assess the relative efficiency of the simultaneous solution procedure of Eq. (6) and the partitioned solution procedure of Eqs. (14–16). For simplicity, the computational efficiency is estimated based on an autonomous, deterministic system, $\boldsymbol{f} = \boldsymbol{w} = \boldsymbol{v} = 0$. The number of floating point operations (FLOP) are approximated to an accuracy of order N, where N is the number of degrees of freedom.

The simultaneous solution procedure given by Eq. (6) may be solved either by a set of first-order differential equation solvers or by a matrix exponential approach. Since the size of the solution matrix is $4N \times 4N$, in addition to factoring of the solution matrix once at the beginning of the computation for time-invariant cases, a total of $16N^2$ storage locations $16N^2$ FLOP are required at each time step.

When one adopts the partitioned solution procedure [Eqs. (14–19)] as S is a sparse and symmetric matrix, considerable storage and computational savings can be attained. If the matrix element bandwidth of S is proportional to N by α

$$\text{bandwidth} = \alpha N; \qquad 0 \leq \alpha \ll N$$

then, only αN^2 storage locations are needed. An L-U decomposition of the S matrix (which requires $([\alpha^2 N^3 + 3\alpha N^2]/2) + O(N)$ FLOP) is performed once for time-invariant systems. Then, at each time step, the RHS is calculated, and a lower and upper triangular solution is performed to compute the states. The partitioned solution procedure requires $[12\alpha N^2 + (r+m)^2 + 4rN + 4mN] + O(N)$ FLOP, where m is the number of actuators and r is the number of measurements. The number of sensors and actuators is usually small compared with the system-order N. The value of α for which the partitioned solution approach becomes more computationally attractive than the simultaneous solution approach must be determined. If the assumption is made that the number of actuators m and the number of measurements r are proportional to N by β

$$r = m = \beta N$$

the values of α and β for which the second-order solution is more efficient than the first-order solution are readily obtained. Figure 4 shows the efficiency ratio of the simultaneous and the partitioned solution procedures for various values of α and β. For practical cases where $\alpha \ll 1$ and $\beta < 1$, the partitioned solution approach is much more efficient.

It should be noted that the FLOP counts assessed previously are only approximate as their precise value depends on the cleverness of the implementation. Also, additional computational savings are possible if more efficient data structures such as profile storage schemes are employed for the mass, damping, and stiffness matrices.

VIII. Numerical Experiments

The time-discretized CSI equations [Eqs. (14–16)] have been implemented into three modules: the structural analyzer, the state estimator, and the solver for the interaction terms. The examples herein have concentrated on the structural analyzer and the control force interaction term. Additional examples with state estimation are presented in Refs. 26 and 28.

The first numerical example is a beam with 10 finite elements and an actuator placed on node 10 as shown in Fig. 5. The actuator gains have been determined using a single mode in the control law. The beam is initially bent according to its first mode shape, and the controller must regulate the response to 1% of its initial amplitude within 0.5 s. Figure 6 illustrates the tip displacement response and the control force $\boldsymbol{u}$ vs time. To assess the effectiveness of the present partitioned solution procedure, the response has been computed using both the simultaneous and the present technique for $N_s = 20$ and $N_s = 10$. The simultaneous technique does not use stabilization [Eq. (9)] for the computation of $\boldsymbol{u}^{n+1/2}$. We have also computed the control force by postprocessing Eq. (1d) instead of the augmented form of Eq. (9). Figure 7 compares the control force error computed by the present stabilized Eq. (12d) vs that by direct substitution into the unaltered Eq. (1d) for step sizes of 20 and 10 samples per response cycle. It is noted that the present partitioned solution procedure yields

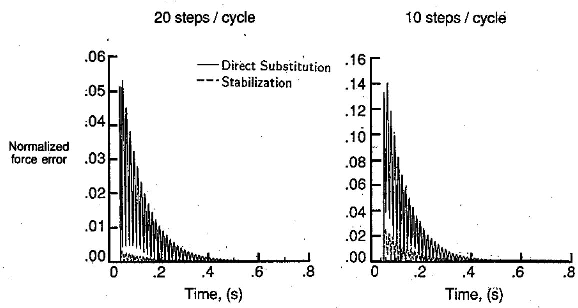

Fig. 7 Errors in computing control force for beam problem.

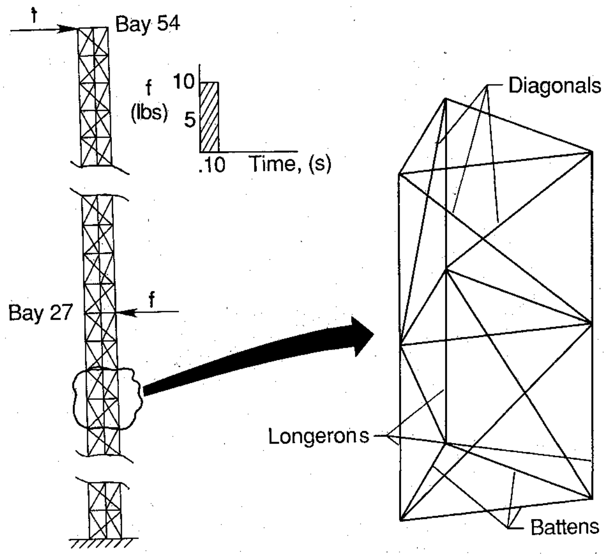

Fig. 8 Truss-beam example.

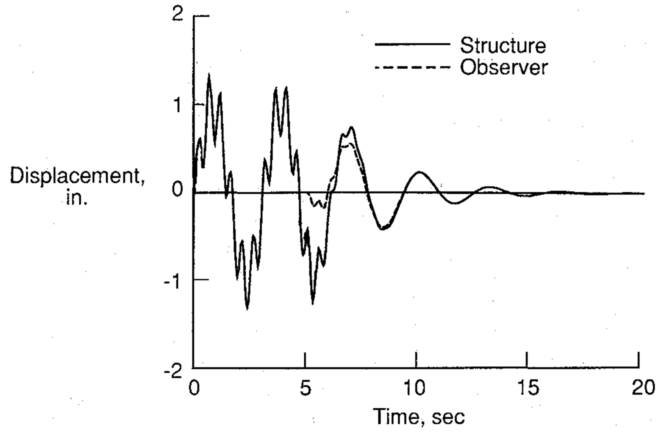

Fig. 9 Truss-beam closed-loop response.

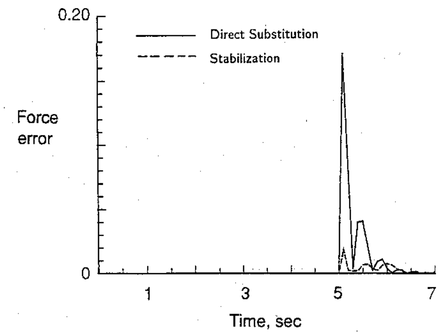

Fig. 10 Truss-beam control force prediction error.

far better accuracy than the conventional direct substitution method.

Of course, one can eliminate the solution errors associated with the conventional simultaneous solution scheme. However, this requires solving the CSI equation in its entirety as given by Eq. (6), which cannot be carried out in a modular software environment.

To illustrate the procedure on more realistic structures, we have performed transient response analyses on the truss beam shown in Fig. 8. The truss was modeled by finite elements with one Timoshenko element from joint to joint. The model had 990 degrees of freedom. The truss is described in more detail in Ref. 25. A modal space control law with seven modes was used to suppress vibtrations of the beam induced by the loading shown in Fig. 8. Both position and rate feedback were used to reduce the vibration amplitude to 0.025 in. within 10 s after active control was initiated.

Figure 9 shows the truss response along with the performance of the second-order observer of Eq. (15). Notice that the observer that is turned on concurrently with the actuator faithfully follows the structural response within an acceptable time lapse. Figure 10 demonstrates the errors in computing the control force by the stabilized Eq. (12d) and by directly substituting into the unaltered Eq. (1d). The computation of the control force by the stabilized Eq. (12d) is usually more than an order of magnitude more accurate than using the original form of Eq. (1d). This high accuracy removes the necessity of iterating during the prediction step.

Most important, the proposed technique, which exploits the second-order form of the structure equation, can be used to solve this 990-degree-of-freedom problem without resorting to a truncated modal model. Thus simulation can be efficiently performed with "truth" models to verify performance and robustness of control laws developed from reduced order models. It is this feature that will alleviate much of the computational difficulties associated with the study of control-structure interactions.

IX. Concluding Remarks

During the development of the present partitioned procedure for conducting control-structure interaction analysis, several theoretical and computational issues have been raised. First, the availability of a second-order observer model is required to exploit the symmetry and sparsity for efficiency purposes. However, second-order observers pose restrictions on the design of observer-based controllers. Second, for time-variant systems, the solution of a Riccati equation to determine the control force with appropriate terminal conditions and the solution of another Riccati equation for the observer equation with given initial conditions can pose bottlenecks in real-time control-structure interaction simulations. (Note that these two solutions are required when using optimal control theory to obtain the control and observer gain matrices.) Parallel computing can potentially remove the aforementioned bottlenecks of the time-variant problem. Finally, we have not considered the intrinsic dynamics of both the actuator and the sensor. However, since the present procedure uses a differential form for the actuator and sensor equations, intrinsic dynamics could be incorporated into the present computational procedure without too much difficulty. Further experiments and inclusions of more realistic control-structure interaction models are necessary before the present simulation procedure can become a production-simulation tool. This is being carried out at present.

Appendix: First-Order Observer Solution

The first-order observer can be numerically integrated as described as follows. Implicit midpoint formulas are used in the time discretization. All variables are defined in Eq. (1).

The first-order observer takes the form

$$\dot{\tilde{x}} = A\tilde{x} + Ef + \bar{B}u + L(z - H\tilde{x}) \tag{A1}$$

Combining like terms, we derive

$$\dot{\tilde{x}} = A_0\tilde{x} + Ef + Lz \tag{A2}$$

where

$$A_0 = \begin{bmatrix} -L_1H_d & I - L_1H_v \\ -M^{-1}(K + BF_1) - L_2H_d & -M^{-1}(D + BF_2) - L_2H_v \end{bmatrix}$$

Time discretization yields

$$(I - \delta A_0)\tilde{x}^{n+1/2} = \delta(Ef^{n+1/2} + Lz^{n+1/2}) + \tilde{x}^n \tag{A3}$$

$$\tilde{x}^{n+1} = 2\tilde{x}^{n+1/2} - \tilde{x}^n \tag{A4}$$

$$u^{n+1/2} = -F\tilde{x}^{n+1/2} \tag{A5}$$

Equation (15) would be replaced by Eq. (A3) in the integration procedure. Then, Eq. (A5) would be used to solve Eq. (16a) and there would be no need to solve for γ. The computational efficiency of the first-order observer of Eq. (A3) is discussed in Sec. VII of the paper.

Acknowledgments

It is a pleasure to acknowledge the support of the first author by the Air Force Office of Scientific Research (AFOSR) through Grant F49620-87-C-0074 and of the second author through a NASA employee fellowship. We thank Anthony K. Amos of AFOSR and Jer-Nan Juang of NASA Langley Research Center for their encouragement during the course of this study.

References

[1]Likins, P. W., "Dynamics and Control of Flexible Space Vehicles," NASA TR-32-1329, 1970.

[2]Balas, M. J., "Active Control of Flexible Systems," *Journal of Optimization Theory and Applications*, Vol. 25, July 1978, pp. 415–436.

[3]Roberson, R. E., "Two Decades of Spacecraft Attitude Control," *Journal of Guidance, Control, and Dynamics*, Vol. 2, No. 1, 1979, pp. 3-8.

[4]Skelton, R., "Theory and Applications in Optimal Control in Aero-space Systems," AGARD Publ. 251, edited by P. Kant, Brussels, Belgium, 1982.

[5]Meirovitch, L., and Silverberg, L. M., "Globally Optimal Control of Self-Adjoint Distributed Systems," *Optimal Control and Applications*, Vol. 4, No. 4, 1983, pp. 365–386.

[6]Horta, L. G., Juang, J. N., and Junkins, J. L., "A Sequential Linear Optimization Approach for Controller Design," AIAA Paper 85-1971, 1985.

[7]Kalman, R. E., "On the General Theory of Control Systems," *Proceedings of the First International Congress on Automatic Control*, Butterworths, London, 1961, pp. 481–491.

[8]Kalman, R. E., and Bucy, R. S., "New Results in Linear Filtering and Prediction Theory," *ASME Journal of Basic Engineering*, Vol. 83, 1961, pp. 95–108.

[9]Kwakernaak, H., and Sivan, R., *Linear Optimal Control Systems*, Wiley-Interscience, New York, 1974.

[10]Arnold, W. F., and Laub, A. J., "Generalized Eigenproblem Algorithms and Software for Algebraic Riccati Equations," *Proceedings of the IEEE*, Vol. 72, No. 12, 1984, pp. 1746–1754.

[11]Park, K. C., "Computational Issues in Control-Structure Interaction Analysis," *Large Space Structures: Dynamics and Control*, edited by S. N. Atluri, Springer-Verlag, Berlin, 1988, pp. 115–131.

[12]Park, K. C., and Belvin, W. K., "Partitioned Procedures for Control-Structure Interaction Analysis," *Computational Mechanics '88*, edited by S. N. Atluri and G. Yagawa, Vol. 2, Springer-Verlag, Berlin, 1988, pp. 64.iii.1–4.

[13]Park, K. C., Felippa, C. A., and DeRuntz, J. A., "Stabilization of Staggered Solution Procedures for Fluid-Structure Interaction Analysis," *Computational Methods for Fluid-Structure Interaction Problems*, edited by T. Belytschko and T. L. Geers, AMD Vol. 26, American Society of Mechanical Engineers, New York, 1977, pp. 95–124.

[14]Park, K. C., and Felippa, C. A., "Partitioned Analysis of Coupled Systems," *Computational Methods for Transient Analysis*, edited by T. Belytschko and T. J. R. Hughes, Elsevier, Amsterdam, 1983, pp. 157–219.

[15]Zienkiewicz, O. C., "Coupled Problems and Their Numerical Solution," *Numerical Methods in Coupled Systems*, edited by R. W. Lewis, P. Bettess, and E. Hinton, Wiley, New York, 1984, pp. 35–38.

[16]Park, K. C., and Chiou, J. C., "Stabilization of Computational Procedures for Constrained Dynamical Systems," *Journal of Guidance, Control, and Dynamics*, Vol. 11, No. 4, 1988, pp. 365–370.

[17]Hughes, P. C., and Skelton, R. E., "Controllability and Observability of Linear Matrix Second-Order Systems," *Journal of Applied Mechanics*, Vol. 47, 1980, pp. 415–420.

[18]Hashemipour, H. R., and Laub, A. J., "Kalman Filtering for Second-Order Models," *Journal of Guidance, Control, and Dynamics*, Vol. 11, No. 2, 1988, pp. 181–185.

[19]Meirovitch, L., and Oz, H., "Modal-Space Control of Distributed Gyroscopic Systems," *Journal of Guidance and Control*, Vol. 5, No. 1, 1982, pp. 140–150.

[20]Yousuff, A., and Skelton, R. E., "Controller Reduction by Component Cost Analysis," *IEEE Transactions on Automatic Control*, Vol. AC-24, 1984, pp. 520–530.

[21]Hale, A. L., Lisowski, R. J., and Dahl, W. E., "Optimal Simultaneous Structural and Control Design of Maneuvering Flexible Spacecraft," *Journal of Guidance, Control, and Dynamics*, Vol. 8, No. 1, 1985, pp. 86–93.

[22]Haftka, R. T., et al., "Sensitivity of Optimized Control Systems to Minor Structural Modifications," 26th Structures, Structural Dynamics and Materials Conference, Orlando, FL, April 1985.

[23]Junkins, J. L., and Rew, D. W., "Unified Optimization of Structures and Controllers," *Large Space Structures: Dynamics and Controls*, edited by S. N. Atluri and A. K. Amos, Springer-Verlag, Berlin, 1988, pp. 323–353.

[24]Khot, N. S., et al., "Optimal Structural Modifications to Enhance the Optimal Active Vibration Control of Large Flexible Structures," 26th Structures, Structural Dynamics, and Materials Conference, Orlando, FL, April 1985.

[25]Belvin, W. K., and Park, K. C., "Structural Tailoring and Feedback Control Synthesis: An Interdisciplinary Approach," AIAA Paper 88-2206, May 1988, also *Journal of Guidance, Control, and Dynamics* (to be published).

[26]Belvin, W. K., and Park, K. C., "On the State Estimation of Structures with Second Order Observers," AIAA Paper 89-1241, April 1989.

[27]Luenberger, D. G., "Observing the State of a Linear System," *IEEE Transactions on Military Electronics*, Vol. 8, 1964, pp. 74–80.

[28]Belvin, W. K., "Simulation and Interdisciplinary Design Methodology for Control-Structure Interaction Systems," Ph.D. Thesis, Center for Space Structures and Controls, Univ. of Colorado, Boulder, CO, Rept. CU-CSSC-89-10, July 1989.

[29]Dahlquist, G., "A Special Stability Problem for Linear Multi-step Methods," *BIT*, Vol. 3, 1963, pp. 27–43.

[30]Hughes, T. J. R., and Belytschko, T., "A Précis of Developments in Computational Methods for Transient Analysis," *Journal of Applied Mechanics*, Vol. 50, 1985, pp. 1033–1041.

[31]Park, K. C., "Transient Analysis Methods in Computational Dynamics," *Finite Elements: Theory and Applications*, edited by D. L. Dwoyer, M. Y. Hussaini, and R. G. Voigt, Springer-Verlag, New York, 1988, pp. 240–267.

[32]Park, K. C., "Partitioned Analysis Procedures for Coupled-Field Problems: Stability Analysis," *Journal of Applied Mechanics*, Vol. 47, 1980, pp. 370–378.

[33]Gantmacher, F. R., *The Theory of Matrices*, Vol. 2, Chelsea, New York, 1959, pp. 190–196.